

THE CATEGORY OF WALDHAUSEN CATEGORIES IS A CLOSED MULTICATEGORY

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INTRODUCTION

The goal of this paper is to develop in detail an example of a closed multicategory. The literature on closed multicategories has very few examples; in this paper we aim to explain a potentially-useful example in enough detail that both the example and the general theory are easier to understand. This paper was written because the details were necessary to the future work of Angelica Osorno and Anna Marie Bohmann, but we hope that it will be useful for others as well. Very little in this paper is new, and it is particularly indebted to [BM11] for many of the ideas.

This paper proves the following theorem:

Theorem 0.1. *The category $\mathbf{WaldCat}$ of Waldhausen categories is a closed symmetric multicategory, in the sense that the hom-sets*

$$\mathbf{WaldCat}_k(A_1, \dots, A_k; B)$$

all have the structure of Waldhausen categories and composition of morphisms is multiexact. In addition, there is a multiexact

$$\mathrm{ev}: \mathcal{C}_1 \times \dots \times \mathcal{C}_k \times \mathbf{WaldCat}_k(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}) \longrightarrow \mathcal{D}$$

defined by

$$(A_1, \dots, A_k, F) \longmapsto F(A_1, \dots, A_k).$$

The organization of this paper is as follows. In Section 1 we introduce multicategories and closed multicategories and in Section 2 we introduce Waldhausen categories. Sections 3 and 4 discuss k -exactness of functors. The definition of the hom-Waldhausen categories is given in Section 5, and the analysis of the K -theory functor is in Section 6.

1. A QUICK INTRODUCTION TO SYMMETRIC MULTICATEGORIES

A multicategory is a generalization of a symmetric monoidal category where one does not necessarily have a product. The motivation for the definition comes from the notion of tensor product: the tensor product of modules classifies bilinear maps out of the ordinary product of the modules. Thus if one is in a context where the tensor product is difficult to work with directly, one can work with bilinear maps instead. The idea of a multicategory is that we have a notion of “ k -linear” map, but we do not necessarily have a representing object, so we must always work with the “ k -linear” maps directly.

More formally, we have the following definition [EM06]:

Definition 1.1. A *symmetric multicategory* \mathcal{M} is given by the following data:

- A collection of objects $\text{ob } \mathcal{M}$.
- For each $k \geq 0$ and $k + 1$ -tuple of objects $A_1, \dots, A_k, B \in \text{ob } \mathcal{M}$, a set $\mathcal{M}_k(A_1, \dots, A_k; B)$ of k -morphisms.
- A right action of Σ_k on the collection of all k -morphisms such that for $\sigma \in \Sigma_k$,

$$\sigma^*: \mathcal{M}_k(A_1, \dots, A_k; B) \longrightarrow \mathcal{M}_k(A_{\sigma(1)}, \dots, A_{\sigma(k)}; B).$$

- A distinguished *unit* $1_A \in \mathcal{M}_1(A; A)$ for every $A \in \text{ob } \mathcal{M}$, and
- A *composition law*

$$\begin{aligned} \circ: \mathcal{M}_\ell(B_1, \dots, B_\ell; C) \times \prod_{i=1}^\ell \mathcal{M}_{k_i}(A_{i1}, \dots, A_{ik_i}; B_i) \\ \longrightarrow \mathcal{M}_{\sum k_i}(A_{11}, \dots, A_{\ell k_\ell}; C). \end{aligned}$$

subject to compatibility axioms listed in [EM06, p5-6]. We do not state them here as we will need to restate them in the enriched setting momentarily.

Any symmetric monoidal category is a symmetric multicategory, by setting

$$\mathcal{M}_k(A_1, \dots, A_k; B) = \mathcal{M}(A_1 \otimes \dots \otimes A_k, B).$$

Thus a symmetric multicategory is the “next best thing” to a symmetric monoidal category in many cases. Many proofs that work in a symmetric monoidal category where the product is defined via an appropriate universal property will also work in a symmetric multicategory.

Quite often one wants more than just a symmetric monoidal category structure: one also wants the symmetric monoidal category to be closed, so that there are hom-objects defined in the category itself. In other words, we want \mathcal{M} to be enriched over itself [Kel82, Section 1.6] (in a way compatible

with the symmetric monoidal structure). We thus have the following definition, where we will write $\overline{A} = (A_1, \dots, A_k)$ in the interest of compactness.

Definition 1.2. A symmetric multicategory \mathcal{M} is called *closed* if for all $k+1$ -tuples A_1, \dots, A_k, B of objects in \mathcal{M} there exists an object $\underline{\mathcal{M}}(\overline{A}; B) \in \mathcal{M}$ with a right Σ_k -action called the *internal hom-object* and an *evaluation morphism*

$$\text{ev}_{\overline{A}; B} \in \mathcal{M}_{k+1}(\overline{A}, \underline{\mathcal{M}}(\overline{A}; B); B).$$

These need to satisfy the following axioms:

(CM1) for all ℓ -tuples $C_1, \dots, C_\ell \in \text{ob } \mathcal{M}$ there is a bijection

$$\varphi_{\overline{A}; \overline{C}; B}: \mathcal{M}_\ell(\overline{C}; \underline{\mathcal{M}}(\overline{A}; B)) \longrightarrow \mathcal{M}_{k+\ell}(\overline{A}, \overline{C}; B)$$

defined by sending $f \in \mathcal{M}_\ell(\overline{C}; \underline{\mathcal{M}}(\overline{A}; B))$ to the composite

$$\text{ev}_{\overline{A}; B} \circ (1_{A_1}, \dots, 1_{A_k}, f).$$

(CM2) This bijection is $\Sigma_k \times \Sigma_\ell$ -equivariant, in the sense that the following diagram commutes for all $(\sigma, \tau) \in \Sigma_k \times \Sigma_\ell$:

$$\begin{array}{ccc} \mathcal{M}_\ell(\overline{C}; \underline{\mathcal{M}}(\overline{A}; B)) & \xrightarrow{\varphi_{\overline{A}; \overline{C}; B}} & \mathcal{M}_{k+\ell}(\overline{A}, \overline{C}; B) \\ f \mapsto \sigma \circ (f \cdot \tau) \downarrow & & \downarrow g \mapsto g \cdot (\sigma \times \tau) \\ \mathcal{M}_\ell(\overline{C}_\tau; \underline{\mathcal{M}}(\overline{A}_\sigma; B)) & \xrightarrow{\varphi_{\overline{A}_\sigma; \overline{C}_\tau; B}} & \mathcal{M}_{k+\ell}(\overline{A}_\sigma, \overline{C}_\tau; B) \end{array}$$

Here $\overline{A}_\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(k)})$ and $\overline{C}_\tau = (C_{\tau(1)}, \dots, C_{\tau(\ell)})$.

For more details, see [Man12, Section 3]; for a more detailed theory of enriched categories, see [Lei04].

For example, if \mathcal{M} is a closed symmetric monoidal category then we can give it the structure of a closed symmetric multicategory by setting $\underline{\mathcal{M}}(A_1, \dots, A_k; B) = B^{A_1 \otimes \dots \otimes A_k}$. Note that if \mathcal{M} is a closed symmetric multicategory then we can think of it as a category enriched over itself.

2. A BIT ABOUT WALDHAUSEN CATEGORIES

We begin by recalling the definition of a Waldhausen category. These were first introduced by Waldhausen in [Wal85], where they are called “categories with cofibrations and weak equivalences.”

Definition 2.1. A *Waldhausen category* is a category \mathcal{C} together with two subcategories, \mathcal{C} and $w\mathcal{C}$, of cofibrations and weak equivalences, satisfying the following extra axioms:

- (W1) All isomorphisms are both weak equivalences and cofibrations.
- (W2) If two of f, g, gf are weak equivalences, so is the third.
- (W3) \mathcal{C} has a zero object, and the morphism $0 \rightarrow A$ is a cofibration for all $A \in \mathcal{C}$.

(W4) Every diagram $C \leftarrow A \hookrightarrow B \in \mathcal{C}$ has a pushout, and the morphism $C \hookrightarrow B \cup_A C$ is a cofibration.

(W5) Given any diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

the induced morphism $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$ is a weak equivalence.

A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between Waldhausen categories \mathcal{C} and \mathcal{D} is 1-exact if it preserves weak equivalences, cofibrations, and pushouts of the form described in (W4).

Before we move on to a very quick overview of the S_\bullet construction for a Waldhausen category, we introduce a couple of technical definitions which will be of great use to us in the upcoming discussion.

Definition 2.2. Let \mathcal{I} be the category with two objects 0 and 1 and one non-invertible morphism $0 \rightarrow 1$. Suppose that we are given two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\alpha: F \rightarrow G$. We write $(F \xRightarrow{\alpha} G)$ for the functor $\mathcal{C} \times \mathcal{I} \rightarrow \mathcal{D}$ given by

$$(F \xRightarrow{\alpha} G)(A) = \begin{cases} F(A) & \text{if } A \in \mathcal{C} \times \{0\} \\ G(A) & \text{if } A \in \mathcal{C} \times \{1\} \end{cases}$$

and

$$(F \xRightarrow{\alpha} G)(f) = \begin{cases} F(f) & \text{if } f \in \mathcal{C} \times \{0\} \\ G(f) & \text{if } f \in \mathcal{C} \times \{1\} \\ \alpha_A & \text{if } f: (A, 0) \rightarrow (A, 1) \end{cases}$$

Note that the existence of such a functor is equivalent to the existence of the natural transformation α .

Definition 2.3. An n -cube in \mathcal{C} is a functor $I: \mathcal{I}^n \rightarrow \mathcal{C}$; a *face* of a cube is a restriction $I|_{\mathcal{I}^k \times \{\epsilon\} \times \mathcal{I}^{n-k-1}}$ for $\epsilon = 0, 1$. We will write the objects of \mathcal{I}^n as vectors $\bar{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$. As shorthand, we will write $I_{k\epsilon}$ for the restriction $I|_{\mathcal{I}^{k-1} \times \{\epsilon\} \times \mathcal{I}^{n-k}}$, for $\epsilon = 0, 1$, and $I(1)$ for $I(1, \dots, 1)$. For any cube I we write I' for the diagram $I|_{\bar{\epsilon} \neq (1, \dots, 1)}$, and define the *southern arrow* of an n -cube I to be the morphism

$$\text{colim } I' \longrightarrow I(1).$$

(The southern arrow may not exist if the colimit does not.) An n -cube I is *good* if its southern arrow is a cofibration and all of its faces are good.

In particular, the 0-cubes are the objects of \mathcal{C} , and the southern arrow of a 0-cube A is just the morphism $\emptyset \rightarrow A$, so all 0-cubes are good. The

1-cubes are the morphisms of \mathcal{C} , and the southern arrow of a 1-cube is itself. Thus the good 1-cubes are the cofibrations.

The notion of a good cube appears in [BM11, Definition 2.1] as a “cubically cofibrant” diagram.

Given a natural transformation $\alpha: I \rightarrow J$ between n -cubes, we will write $[\alpha]$ for the $n+1$ -cube $(I \xrightarrow{\alpha} J)$.

Let $\langle n \rangle$ be the ordered set $0 < 1 < \dots < n$ and let $\text{Ar}\langle n \rangle$ be the arrow category of $\langle n \rangle$; we will denote an object in $\text{Ar}\langle n \rangle$ as $j < i$. For a vector $\vec{n} = (n_1, \dots, n_m)$ we will write $\langle \vec{n} \rangle = \langle n_1 \rangle \times \dots \times \langle n_m \rangle$.

Definition 2.4. The category $S_n\mathcal{C}$ is defined to have as objects functors $X: \text{Ar}\langle n \rangle \rightarrow \mathcal{C}$ satisfying the extra conditions

- (1) $X(i = i) = *$ for all $i \in \langle n \rangle$, and
- (2) $X(i < j) \rightarrow X(i < k)$ is a cofibration, and
- (3) for all $i < j < k$ the square

$$\begin{array}{ccc} X(i < j) & \hookrightarrow & X(i < k) \\ \downarrow & & \downarrow \\ X(j = j) & \hookrightarrow & X(j < k) \end{array}$$

is a pushout square.

The categories $S_n\mathcal{C}$ form a simplicial category by inheriting the simplicial structure from the simplicial category $[\text{Ar}\langle \cdot \rangle, \mathcal{C}]$.

$S_n\mathcal{C}$ is defined to be a Waldhausen category by setting the weak equivalences to be levelwise, and the cofibrations to be the natural transformations $\alpha: X \rightarrow Y$ such that for all $i < j$ the square

$$\begin{array}{ccc} X(i) & \hookrightarrow & X(j) \\ \downarrow & & \downarrow \\ Y(i) & \hookrightarrow & Y(j) \end{array}$$

is good.

As applying S to a Waldhausen category produces a simplicial Waldhausen category we can iterate the construction. It is not very difficult to see that the k -fold iterated construction $S^{(k)}\mathcal{C}$ has objects which are functors

$$X: \text{Ar}(\langle n_1 \rangle \times \dots \times \langle n_k \rangle) \rightarrow \mathcal{C},$$

such that for every k -cube $I: \mathcal{I}^k \rightarrow \text{Ar}\langle \vec{n} \rangle$, the k -cube $X \circ I$ is good, and has as cofibrations the natural transformations $\alpha: X \rightarrow Y$ such that the $k+1$ -cube $(X \circ I \xrightarrow{\alpha} Y \circ I)$ is good.

Definition 2.5. Let \mathbf{Sp} be the category of symmetric spectra. The functor $K: \mathbf{WaldCat} \rightarrow \mathbf{Sp}$ is defined by taking a Waldhausen category \mathcal{C} to the symmetric spectrum

$$(|w\mathcal{C}|, |wS\mathcal{C}|, |wS^{(2)}\mathcal{C}|, \dots).$$

For more details, see [Wal85]; for more on an all-at-once construction of $K(\mathcal{C})$, see [BM11, Section 2].

Definition 2.6. A functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ of Waldhausen categories is *biexact* if the following conditions hold.

- (1) For any object $A \in \mathcal{C}$, $F(A, -)$ is exact; for any object $B \in \mathcal{D}$, $F(-, B)$ is exact.
- (2) For any two cofibrations $f: A \hookrightarrow A' \in \mathcal{C}$ and $g: B \hookrightarrow B' \in \mathcal{D}$, the southern arrow of the square

$$\begin{array}{ccc} F(A, B) & \xrightarrow{F(f, 1_B)} & F(A', B) \\ F(1_A, g) \downarrow & & \downarrow F(1_{A'}, g) \\ F(A, B') & \xrightarrow{F(f, 1_{B'})} & F(A', B') \end{array}$$

is a cofibration.

The definition of biexact functor is meant to be analogous to the definition of bilinear map. If \mathcal{C} , \mathcal{D} and \mathcal{E} are all equal then this should correspond to a product structure on $K(\mathcal{C})$. In an ideal situation, $\mathbf{WaldCat}$ would have a monoidal structure \otimes representing biexact functors, and all we would need to show is that K is symmetric monoidal. Unfortunately, this cannot happen:

Proposition 2.7. *There does not exist a symmetric monoidal product \otimes on $\mathbf{WaldCat}$ such that*

$$\left\{ \begin{array}{c} \text{exact functors} \\ \mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{E} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{biexact functors} \\ \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E} \end{array} \right\}.$$

This result is well-known to experts, but as we could not find a reference for it we present a proof here.

Proof. Let \mathbf{N}_* be the full subcategory of \mathbf{FinSet}_* with objects the finite pointed sets $\underline{n} \stackrel{\text{def}}{=} \{*, 1, \dots, n\}$ for $n \geq 0$; note that \mathbf{N}_* is equivalent to \mathbf{FinSet}_* . As all Waldhausen categories contain all binary coproducts, any Waldhausen category contains \mathbf{N}_* as a Waldhausen subcategory. Suppose that such a symmetric monoidal structure on $\mathbf{WaldCat}$ exists, and let \mathcal{S} be the unit. Then by assumption, $\mathcal{S} \otimes \mathbf{N}_* \cong \mathbf{N}_*$ and the set of exact functors $\mathbf{N}_* \rightarrow \mathbf{FinSet}_*$ is given by a choice of sets (A_1, \dots, A_n, \dots) such that $|A_n| = n|A_1|$.

Let $\iota: \mathbf{N}_* \rightarrow \mathcal{S}$ be any inclusion of \mathbf{N}_* as a subcategory of \mathcal{S} , and let $F: \mathcal{S} \times \mathbf{N}_* \rightarrow \mathbf{FinSet}_*$ be any biexact functor. By assumption, F must be uniquely determined by a choice of sets (A_1, \dots) . However, unlike in \mathbf{N}_* , in

$\mathcal{S} \times \mathbf{N}_*$ there are multiple objects whose image under F must have the same cardinality; for example,

$$|F(\iota(\underline{1}) \amalg \iota(\underline{1}), \underline{1})| = |F(\iota(\underline{1}), \underline{1} \amalg \underline{1})|.$$

Thus a single set A_n cannot determine the value of F , and we see that such a bijection cannot exist. \square

We want to show that even though **WaldCat** does not have a symmetric monoidal structure, it does have the next best thing: a symmetric multicategory structure where the 2-ary morphisms are exactly the biexact functors.

3. CUBES

In this section we develop some technical aspects of the theory of cubes. The category \mathcal{C} will always be assumed to be a Waldhausen category.

The general idea of this section is that good n -cubes should behave like objects in a Waldhausen category, and that cofibrations between them should correspond to good $n+1$ -cubes. More formally, we have the following proposition, which is designed to be a higher-dimensional analog of Axiom (W4).

Proposition 3.1. *Let I , J and K be good n -cubes in \mathcal{C} , and suppose that $\alpha: I \Rightarrow J$ is a natural transformation. If $[\alpha]$ is a good $n+1$ -cube then the diagram*

$$K \longleftarrow I \xRightarrow{\alpha} J$$

has a pushout $J \cup_I K$, and the natural transformation $\beta: K \Rightarrow J \cup_I K$ gives a good $n+1$ -cube $[\beta]$.

Note that as $J \cup_I K$ is a face of $[\beta]$ it must be good as well.

The proof of this proposition is quite long, so to aid understanding we first spend some time developing a general theory of cubes. The first result we mention is the $n = 1$ case of the proposition, which is proved as lemma 1.1.1 in [Wal85]:

Lemma 3.2. *Consider the diagram*

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ C' & \longleftarrow & A' & \longrightarrow & B' \end{array}$$

in \mathcal{C} . If the right-hand square is good (as a 2-cube) then the induced morphism $B \cup_A C \hookrightarrow B' \cup_{A'} C'$ is a cofibration.

The next couple of lemmas are general category-theoretic observations whose proofs are simple, but which we will need several times in the forthcoming proofs.

Lemma 3.3. *If \mathcal{D} is any category with a terminal object $*$ and $F: \mathcal{A} \times \mathcal{D} \rightarrow \mathcal{C}$ is a functor, then*

$$\operatorname{colim} F \cong \operatorname{colim} F|_{\mathcal{A} \times \{*\}}.$$

Proof. This follows because the functor $\mathcal{A} \times \{*\} \rightarrow \mathcal{A} \times \mathcal{D}$ is cofinal. \square

Many of our proofs rely on computing southern arrows of cubes; luckily, it turns out that these can be deduced from simple pushouts. The following lemma is an n -dimensional generalization of a special case of the rigid dual of Proposition 0.2 in [Goo92]; we state it here as we will be using it several times in this section. We defer the proof until Appendix A.

Lemma 3.4. *Suppose that we are given an indexing category \mathcal{A} along with n subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$. Let $F: \mathcal{A} \rightarrow \mathcal{C}$. Then the southern arrow of the cube I given by*

$$I(\bar{\epsilon}) = \begin{cases} \operatorname{colim} F|_{\bigcap_{\epsilon_i=0} \mathcal{A}_i} & \bar{\epsilon} \neq (1, \dots, 1) \\ \operatorname{colim} F & \bar{\epsilon} = (1, \dots, 1) \end{cases}$$

is an isomorphism.

The special case that we will use the most often is the following. Let $I: \mathcal{I}^n \rightarrow \mathcal{C}$ be an n -cube. Set $\mathcal{A} = \mathcal{I}^n \setminus \{(1, \dots, 1)\}$, $\mathcal{A}_1 = \mathcal{A} \setminus \{(1, \dots, 1, 0)\}$, and $\mathcal{A}_2 = \mathcal{I}^{n-1} \times \{0\}$. Then

$$\begin{aligned} \operatorname{colim} I|_{\mathcal{A}_1} &= \operatorname{colim} (I_{n1})' \\ \operatorname{colim} I|_{\mathcal{A}_2} &\cong I(1, \dots, 1, 0) \\ \operatorname{colim} I|_{\mathcal{A}_1 \cap \mathcal{A}_2} &= \operatorname{colim} (I_{n0})'. \end{aligned}$$

Applying Lemma 3.4 we have a pushout square

$$\begin{array}{ccc} \operatorname{colim} (I_{n0})' & \longrightarrow & I(1, \dots, 1, 0) \\ \downarrow & & \downarrow \\ \operatorname{colim} (I_{n1})' & \longrightarrow & \operatorname{colim} I' \end{array}$$

which will allow us to compute the southern arrow of I using pushouts and induction.

We now turn to the existence of southern arrows.

Lemma 3.5. *Let I be a good n -cube in a Waldhausen category \mathcal{C} . Then $\operatorname{colim} I'_{n0} \rightarrow \operatorname{colim} I'_{n1}$ is a cofibration, and the southern arrow of I exists.*

Proof. We prove that $\operatorname{colim} I'_{n0} \rightarrow \operatorname{colim} I'_{n1}$ is a cofibration by induction on n . The cases $n = 1, 2$ follow directly from the definition of a good n -cube; the case $n = 3$ is a special case of Lemma 3.2. Now suppose that this is true for all cubes of size less than n , and consider the situation for n -cubes.

Let $A = I(0, \dots, 0)$, $B = I(0, \dots, 0, 1)$, $X = I(1, \dots, 1, 0, 0)$ and $Y = I(1, \dots, 1, 0, 1)$. Then (by Lemma 3.4) we know that

$$\operatorname{colim} I'_{n0} \cong \operatorname{colim} \left(\operatorname{colim}((I_{n0})_{(n-1)1})' \longleftarrow A \hookrightarrow X \right)$$

and

$$\operatorname{colim} I'_{n1} \cong \operatorname{colim} \left(\operatorname{colim}(I_{n1})_{(n-1)1})' \longleftarrow B \hookrightarrow Y \right).$$

Note that $(I_{n0})_{(n-1)1} = (I_{(n-1)1})_{(n-1)0}$ and $(I_{n1})_{(n-1)1} = (I_{(n-1)1})_{(n-1)1}$, so the inductive hypothesis applies to these and we get a diagram

$$\begin{array}{ccccc} \operatorname{colim}((I_{n0})_{(n-1)1})' & \longleftarrow & A & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{colim}((I_{n1})_{(n-1)1})' & \longleftarrow & B & \hookrightarrow & Y \end{array}$$

As I is good the right-hand square is also good, and thus we see that Lemma 3.2 applies and the induced morphism between the pushouts is a cofibration, as desired.

In order for the southern arrow to exist we need to show that $\operatorname{colim} I'$ exists. By Lemma 3.4 we know that

$$\begin{aligned} \operatorname{colim} I' &\cong \operatorname{colim} \left(\operatorname{colim} I_{n0} \longleftarrow \operatorname{colim} I'_{n0} \longrightarrow \operatorname{colim} I|_{\bar{\epsilon} \notin \{1\}^{n-1} \times \mathcal{I}} \right) \\ &\cong \operatorname{colim} \left(I_{n0}(1) \longleftarrow \operatorname{colim} I'_{n0} \longrightarrow \operatorname{colim} I'_{n1} \right), \end{aligned}$$

where $\operatorname{colim} I|_{\bar{\epsilon} \notin \{1\}^{n-1} \times \mathcal{I}} \cong \operatorname{colim} I'_{n1}$ by Lemma 3.3. But by the first part of the proof the second morphism in the colimit is a cofibration, so the pushout exists. \square

As a consequence we get the following:

Lemma 3.6. *In a pushout square of n -cubes*

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & J \\ \downarrow & & \downarrow \\ K & \xrightarrow{\beta} & J \cup_I K \end{array}$$

where I , J , K and $[\alpha]$ are good, the southern arrows of $J \cup_I K$ and $[\beta]$ exist.

Proof. As colimits commute we know that

$$\operatorname{colim} (J \cup_I K)' \cong \operatorname{colim} \left(\operatorname{colim} K' \longleftarrow \operatorname{colim} I' \xrightarrow{i} \operatorname{colim} J' \right),$$

where i is a cofibration by Lemma 3.5 because $[\alpha]$ is good. Each term in the pushout on the right exists because I , J and K are good, and the pushout itself exists because i is a cofibration; thus the southern arrow of $J \cup_I K$ exists.

As K is good we know that its southern arrow is a cofibration, and by the special case of Lemma 3.4 discussed on page 8 the southern arrow of $[\beta]$ exists if the southern arrow of $[\beta]_{(n+1)1}$ exists. But $[\beta]_{(n+1)1}$ is exactly $J \cup_I K$ which has a southern arrow, as desired. \square

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Note that I gives us a natural transformation of functors from I' to the constant functor at $I(1)$.

We will prove the desired statement by induction on n . Clearly for $n = 0$ we just need to show that cofibrations are preserved under pushout in \mathcal{C} , which we know is true because \mathcal{C} is a Waldhausen category. For $n = 1$ this is just Lemma 3.2. Thus assume that $n > 1$ and that we know that the proposition holds for all smaller dimensions. By restricting to faces of I, J, K and using the inductive hypothesis we see that any face of $[\beta]$ that is not equal to K or $J \cup_I K$ is good. K is given to be good, so in order to prove the proposition it suffices to show that the southern arrows of $J \cup_I K$ and $[\beta]$ are cofibrations; we know that they exist by Lemmas 3.5 and 3.6.

The southern arrow of β is the morphism

$$j: \operatorname{colim} \left((J \cup_I K)' \xleftarrow{\beta} K' \Rightarrow K(1) \right) \longrightarrow (J \cup_I K)(1);$$

we want to show that it is a cofibration. Note that

$$\begin{aligned} \operatorname{colim} \quad & \begin{array}{ccc} K' & \Longrightarrow & K(1) \\ \beta \Downarrow & & \\ (J \cup_I K)' & & \end{array} & \cong \operatorname{colim} \quad \begin{array}{ccccc} I' & \Longrightarrow & K' & \Longrightarrow & K(1) \\ \alpha \Downarrow & & \Downarrow \beta & & \\ J' & \Longrightarrow & (J \cup_I K)' & & \end{array} \\ & \cong \operatorname{colim} (J' \xleftarrow{\alpha} I' \Longrightarrow K(1)) \\ & \cong \operatorname{colim} (J' \xleftarrow{\alpha} I' \Longrightarrow I(1) \longrightarrow K(1)) \end{aligned}$$

where the second line follows because the square on the first line is a pushout square. Therefore we have the following diagram, where the three commutative squares are all pushout squares:

$$\begin{array}{ccccc} I' & \Longrightarrow & I(1) & \longrightarrow & K(1) \\ \alpha \Downarrow & & \downarrow & & \downarrow \\ J' & \Longrightarrow & \operatorname{colim} [\alpha]' & \longrightarrow & K(1) \cup_{I(1)} \operatorname{colim} [\alpha] \\ & \searrow & \downarrow i & & \downarrow j \\ & & J(1) & \longrightarrow & (J \cup_I K)(1) \end{array}$$

Since $[\alpha]$ is a good cube it follows that i is a cofibration, and thus j is a cofibration because it is a pushout of i .

It now remains to show that the southern arrow of $J \cup_I K$ is a cofibration. The southern arrow of $J \cup_I K$ factors through the southern arrow of $[\beta]$; thus it suffices to show that the connecting morphism is also a cofibration. By Lemma 3.4 we know that

$$\begin{aligned} \operatorname{colim}[\beta]' &\cong \operatorname{colim} \left(\operatorname{colim} K \longleftarrow \operatorname{colim} K' \longrightarrow \operatorname{colim} [\beta] \Big|_{\bar{\epsilon} \notin \{1\}^k \times \mathcal{I}} \right) \\ &\cong \operatorname{colim} \left(K(1) \longleftarrow \operatorname{colim} K' \longrightarrow \operatorname{colim} (J \cup_I K)' \right) \end{aligned}$$

where the left-hand map on the bottom is a cofibration because K is a good cube, and $\operatorname{colim}[\beta] \Big|_{\bar{\epsilon} \notin \{1\}^n \times \mathcal{I}} \cong \operatorname{colim}(J \cup_I K)'$ by Lemma 3.3. The induced morphism $\operatorname{colim}(J \cup_I K)' \rightarrow \operatorname{colim}[\beta]'$ is the right-hand map in the given pushout square. Given that it is the pushout of the cofibration $\operatorname{colim} K' \hookrightarrow K(1)$ we know that it is also a cofibration, as desired. Thus $J \cup_I K$ is a good cube, and we are done. \square

4. k -EXACTNESS

The goal of this section is to show that **WaldCat** is a symmetric multicategory, where **WaldCat** $(\mathcal{C}, \mathcal{D})$ is the set of exact functors $\mathcal{C} \rightarrow \mathcal{D}$ and **WaldCat** $_k(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})$ is the set of k -exact functors. Most of this section is devoted to defining k -exactness and working out its properties.

When k is clear from context, we will write $\bar{\mathcal{C}} = \mathcal{C}_1 \times \dots \times \mathcal{C}_k$, and refer to objects $\bar{A} = (A_1, \dots, A_k) \in \bar{\mathcal{C}}$, and $\bar{f} = (f_1, \dots, f_k) \in \bar{\mathcal{C}}$.

Definition 4.1. Given morphisms $f_i: A_{i0} \rightarrow A_{i1} \in \mathcal{C}_i$ and a functor $F: \bar{\mathcal{C}} \rightarrow \mathcal{D}$ we define a k -cube $[\bar{f}]_F$ in \mathcal{D} by

$$[\bar{f}]_F(\epsilon_1, \dots, \epsilon_k) = F(A_{1\epsilon_1}, \dots, A_{k\epsilon_k})$$

and

$$[\bar{f}]_F(\epsilon_1 \rightarrow \epsilon'_1, \dots, \epsilon_k \rightarrow \epsilon'_k) = F(h_1, \dots, h_k),$$

where $h_i = 1_{A_{i\epsilon_i}}$ if $\epsilon_i = \epsilon'_i$ and $h_i = f_i$ otherwise. We define the *box product* of \bar{f} to be the southern arrow of $[\bar{f}]_F$.

Definition 4.2. A 0-exact functor with codomain \mathcal{D} is an object of \mathcal{D} . A functor $F: \mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}$ is k -exact if

- (kE1) $F(\bar{A}) = 0$ if $A_i = 0$ for any $1 \leq i \leq k$,
- (kE2) F preserves pushouts in each variable independently,
- (kE3) $F(w\bar{\mathcal{C}}) \subseteq w\mathcal{D}$, and
- (kE4) for all $\bar{f} \in \bar{\mathcal{C}}$, $[\bar{f}]_F$ is good.

If a functor is k -exact for some k , we will call it *multiexact*.

Definition 4.3. Let **WaldCat** be the following symmetric multicategory:

- objects:** Waldhausen categories.
- k -morphisms:** k -exact functors $\mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}$.
- Σ_k -action:** permuting the input variables.

To check that this is well-defined it suffices to check that multiexact functors compose properly. This is exactly the result of Proposition 4.7; however, before we can prove that we need to develop a little bit of the theory of k -exact functors. The first lemma we prove shows that in order to prove Axiom (kE4) it suffices to consider a single type of southern arrow.

Lemma 4.4. *Let $F: \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \longrightarrow \mathcal{D}$ be any functor satisfying Axiom (kE1). If the southern arrow of $[\bar{f}]_F$ is a cofibration for all $\bar{f} \in \mathcal{C}$ then F satisfies Axiom (kE4).*

Proof. Fix $\bar{f} \in \mathcal{C}$, writing $f_i: A_{i0} \longrightarrow A_{i1}$. We know that the southern arrow of $[\bar{f}]_F$ is a cofibration, so all that we need to show is that all faces of $[\bar{f}]_F$ are good. Let $J \subseteq \{1, \dots, n\}$ and let $\bar{\eta} \in \{0, 1\}^k$. A face of $[\bar{f}]_F$ is determined by J and $\bar{\eta}$ by considering the cube $I_{J, \bar{\eta}}: \mathcal{I}^{|J|} \longrightarrow \mathcal{D}$ given by

$$I_{J, \bar{\eta}}(\epsilon_{j_1}, \dots, \epsilon_{j_{|J|}}) = F(B_1, \dots, B_k) \quad B_j = \begin{cases} A_{j\eta_j} & \text{if } j \notin J \\ A_{j\epsilon_j} & \text{if } j \in J. \end{cases}$$

More informally, we let J determine which variables are allowed to change, and let $\bar{\eta}$ determine the value of the other variables. Let \bar{h} have $h_j = f_j$ if $j \in J$ and $0 \longrightarrow A_{j\eta_j}$ otherwise. Then $\bar{h} \in \mathcal{C}$, so by assumption we know that the southern arrow of $[\bar{h}]_F$ is a cofibration. But this southern arrow is exactly the southern arrow of $I_{J, \bar{\eta}}$, so we see that the southern arrow of $I_{J, \bar{\eta}}$ is a cofibration, as desired. \square

We now need to consider composition of multiexact functors. First, a simple observation about southern arrows.

Lemma 4.5. *A k -exact functor commutes with taking southern arrows in each variable independently.*

Proof. This follows from Axiom (kE2) and Lemma 3.4, which says that southern arrows can be computed as iterated pushouts. \square

The next two results concern the following situation. Let $j_1, \dots, j_k \in \mathbb{Z}_{\geq 0}$, and write $m_i = \sum_{\ell=1}^i j_\ell$. Given j_i -exact functors

$$G_i: \mathcal{C}_{m_{i-1}+1} \times \cdots \times \mathcal{C}_{m_i} \longrightarrow \mathcal{D}_i$$

and a k -exact functor

$$F: \mathcal{D}_1 \times \cdots \times \mathcal{D}_k \longrightarrow \mathcal{D},$$

define the composite functor

$$H = F \circ (G_1 \times \cdots \times G_k): \mathcal{C}_1 \times \cdots \times \mathcal{C}_{m_k} \longrightarrow \mathcal{D}.$$

Lemma 4.6. *Let g_i be the southern arrow of $[(f_{m_{i-1}+1}, \dots, f_{m_i})]_{G_i}$. The southern arrow of $[\bar{f}]_H$ is isomorphic to the southern arrow of $[\bar{g}]_F$.*

Proof. We will use Lemma 3.4. Let $\mathcal{A}_i = \mathcal{I}^{m_{i-1}} \times (\mathcal{I}^{j_i} \setminus \{(1, \dots, 1)\}) \times \mathcal{I}^{m_k - m_i}$, so that for any $J \subseteq \{1, \dots, k\}$,

$$\bigcap_{j \in J} \mathcal{A}_j = \prod_{i=1}^k B_i^{(J)}, \quad \text{where } B_i^{(J)} = \begin{cases} \mathcal{I}^{j_i} & \text{if } i \in J \\ \mathcal{I}^{j_i} \setminus \{(1, \dots, 1)\} & \text{if } i \notin J \end{cases}$$

and

$$\bigcup_{i=1}^n \mathcal{A}_i = \mathcal{I}^{m_k} \setminus \{(1, \dots, 1)\}.$$

In addition, by Lemma 4.5

$$\text{colim } F \circ (G_1 \times \dots \times G_k) \big|_{\bigcap_{j \in J} \mathcal{A}_j} \cong F \left(\text{colim } G_1 \big|_{B_1^{(J)}}, \dots, \text{colim } G_k \big|_{B_k^{(J)}} \right).$$

Note that these are exactly the entries of the cube $[\overline{g}]_F$. Applying Lemma 3.3 to compute $\text{colim } [\overline{f}]'_H$ in terms of these, we see that the southern arrow of $[\overline{f}]_H$ is exactly the southern arrow of $[\overline{g}]_F$, as desired. \square

Using this we can show that H is m_k -exact, and thus that **WaldCat** is a multicategory.

Proposition 4.7. *WaldCat is a multicategory.*

Proof. It suffices to check that H is m_k -exact. Axioms (kE1), (kE2) and (kE3) are direct from the definition, so we just need to check Axiom (kE4).

Let $\overline{f} \in \overline{\mathcal{C}}$; we want to show that $[\overline{f}]_H$ is good. By Lemma 4.4 it suffices to show that the southern arrow of $[\overline{f}]_H$ is a cofibration. Let g_i be the southern arrow of $[(f_{m_{i-1}+1}, \dots, f_{m_i})]_{G_i}$. As G_i is j_i -exact we know that g_i is a cofibration; as F is k -exact, the southern arrow of $[\overline{g}]_F$ is also a cofibration. However, by Lemma 4.6 we know that the southern arrow of $[\overline{g}]_F$ is exactly the southern arrow of $[\overline{f}]_H$, so we see that the southern arrow of $[\overline{f}]_H$ must also be a cofibration, as desired. \square

5. THE CLOSED STRUCTURE

We would now like to show that **WaldCat** is a closed multicategory.

Definition 5.1. We define the internal hom $\underline{\mathbf{WaldCat}}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})$ in the following manner:

- objects:** k -exact functors $\mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{D}$,
- morphisms:** natural transformations between functors,
- weak equivalences:** natural weak equivalences between functors, and
- cofibrations:** natural transformations $\alpha: F \rightarrow G$ such that for any $\overline{f} \in \overline{\mathcal{C}}$, the cube

$$([\overline{f}]_F \xRightarrow{\alpha} [\overline{f}]_G)$$

is good.

In particular, note that all cofibrations are levelwise cofibrations. We need to prove that this is well-defined.

Lemma 5.2. $\mathbf{WaldCat}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})$ is a Waldhausen category.

Proof. The only axiom that is nontrivial to check is Axiom (W4); the others follow directly from the fact that \mathcal{D} is a Waldhausen category. Thus we focus our attention on checking Axiom (W4).

Consider a diagram

$$H \longleftarrow F \xrightarrow{\alpha} G \in \mathbf{WaldCat}(\overline{\mathcal{C}}; \mathcal{D}).$$

We know that α is a levelwise cofibration and that pushouts along cofibrations exist in \mathcal{D} , so we get a functor $G \cup_F H: \overline{\mathcal{C}} \rightarrow \mathcal{D}$. We need to check that this functor is k -exact, and that the induced natural transformation $\beta: H \rightarrow G \cup_F H$ is a cofibration inside $\mathbf{WaldCat}(\overline{\mathcal{C}}; \mathcal{D})$.

The functor $G \cup_F H$ satisfies Axiom (kE1) and (kE2) because F , G and H are k -exact, and has Axiom (kE3) because Axiom (W5) holds in \mathcal{D} . To prove Axiom (kE4), fix $\overline{f} \in \overline{\mathcal{C}}$. We know that $[\overline{f}]_F$, $[\overline{f}]_G$ and $[\overline{f}]_H$ are good, so $[\overline{f}]_{G \cup_F H}$ is also good by Proposition 3.1. Thus $G \cup_F H$ is k -exact.

Proposition 3.1 also tells us that $([\overline{f}]_H \xrightarrow{\beta} [\overline{f}]_{G \cup_F H})$ is good, which means that β is a cofibration, as desired. \square

In order for this definition to make $\mathbf{WaldCat}$ into a closed multicategory, we need a $k + 1$ -exact evaluation morphism.

Definition 5.3. The functor

$$\mathrm{ev}_{\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}}: \mathcal{C}_1 \times \dots \times \mathcal{C}_k \times \mathbf{WaldCat}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}) \longrightarrow \mathcal{D}$$

is defined by

$$\mathrm{ev}_{\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}}(A_1, \dots, A_k, F) = F(A_1, \dots, A_k).$$

Lemma 5.4. The functor $\mathrm{ev}_{\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}}$ is $k + 1$ -exact.

Proof. Axioms (kE1), (kE2) and (kE3) are easily checked from the fact that F is k -exact. Thus we only need to check Axiom (kE4). Given $f_i: A_i \hookrightarrow B_i \in \mathcal{C}_i$ and a cofibration $\alpha: F \rightarrow G$ we need to check that $[(\overline{f}, \alpha)]_{\mathrm{ev}}$ is good.

However, by definition this cube is the cube $([\overline{f}]_F \xrightarrow{\alpha} [\overline{f}]_G)$, which is good because F and G are k -exact and α is a cofibration. \square

It remains to check that this gives a well-defined closed multicategory structure on $\mathbf{WaldCat}$, i.e. that for all Waldhausen categories $\mathcal{C}_1, \dots, \mathcal{C}_k$, \mathcal{D} and $\mathcal{A}_1, \dots, \mathcal{A}_\ell$ the function

$$\begin{aligned} & \mathbf{WaldCat}_\ell(\mathcal{A}_1, \dots, \mathcal{A}_\ell; \mathbf{WaldCat}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})) \\ & \longrightarrow \mathbf{WaldCat}_{k+\ell}(\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{A}_1, \dots, \mathcal{A}_\ell; \mathcal{D}) \end{aligned}$$

given by $F \mapsto \mathrm{ev}_{\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D}} \circ (1_{\mathcal{C}_1}, \dots, 1_{\mathcal{C}_k}, F)$ is a bijection. We can construct an inverse easily by partial application, so assuming that the partial inverse is well-defined we know that this is a bijection. It is also clearly $\Sigma_k \times \Sigma_\ell$ -equivariant. Thus we need to show the following:

Lemma 5.5. *Fix $1 \leq \ell \leq k$. Let $F: \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \rightarrow \mathcal{D}$ be a k -exact functor, and fix $A_i \in \mathcal{C}_i$ for $\ell < i \leq k$. Then the functor*

$$F(-, A_{\ell+1}, \dots, A_k): \mathcal{C}_1 \times \cdots \times \mathcal{C}_\ell \longrightarrow \mathcal{D}$$

is ℓ -exact.

Proof. It suffices to prove this for $\ell = k - 1$; the rest will follow by induction. Axioms (kE1), (kE2) and (kE3) hold immediately, so it suffices to consider (kE4). Let $f_i \in \mathcal{C}_i$ for $1 \leq i < k$ be cofibrations, and let $f_k: \emptyset \hookrightarrow A_k$ for the A_k in the statement of the lemma. Since F was k -exact, we know that $[f]_F$, which implies that $([f]_F)_{k1}$ is good. But this is exactly the cube $[(f_2, \dots, f_k)]_{F(-, A_k)}$. Thus $F(-, A_k)$ is $k - 1$ -exact, as desired. \square

We have now proved:

Proposition 5.6. *WaldCat is a closed multicategory.*

6. K -THEORY AS AN ENRICHED MULTIFUNCTOR

Our goal for this section is to show that the closed multicategory structure on **WaldCat** is compatible with the K -theory functor.

Definition 6.1. A multifunctor $F: \mathcal{M} \rightarrow \mathcal{M}'$ between symmetric multicategories is a function $F: \text{ob } \mathcal{M} \rightarrow \text{ob } \mathcal{M}'$, and a function

$$\mathcal{M}(A_1, \dots, A_k; B) \longrightarrow \mathcal{M}'(F(A_1), \dots, F(A_k); F(B))$$

for all tuples B, A_1, \dots, A_k of objects. These must preserve the units and be compatible with composition and the Σ_k -action. In the case when \mathcal{M} and \mathcal{M}' are enriched over a symmetric monoidal \mathcal{V} , we just need

$$\mathcal{M}(A_1, \dots, A_k; B) \longrightarrow \mathcal{M}'(F(A_1), \dots, F(A_k); F(B))$$

to be a \mathcal{V} -morphism.

First we consider the unenriched setting.

Proposition 6.2. *The functor $K: \mathbf{WaldCat} \rightarrow \mathbf{Sp}$ is a multifunctor.*

This statement is well-known to specialists, and has been mentioned in many papers, including [Wal85], [BM11, Theorem 2.6], and [GH99]. However, we could not find a reference that explicitly checked that the multifunctor structure was compatible with the structure maps of the symmetric spectra produced by K -theory, so in the interest of completeness we present the proof here, as well.

Before starting the proof of this proposition we will first make an auxiliary construction. In order to show that K is a multifunctor we will need to show that any k -exact functor $F: \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \rightarrow \mathcal{C}$ gives rise to a morphism $K(\mathcal{C}_1) \wedge \cdots \wedge K(\mathcal{C}_k) \rightarrow K(\mathcal{C})$. When $k = 0$ this says that a choice of object $A \in \mathcal{C}$ gives a morphism $\mathbb{S} \rightarrow K(\mathcal{C})$. To construct this, choose an equivalence $\alpha: \mathbb{S} \rightarrow K(\mathbf{FinSet}_*)$, and then take the exact functor $p_A: \mathbf{FinSet}_* \rightarrow \mathcal{C}$ given by $I \mapsto \coprod_I A$. Then $K(p_A)\alpha$ is the desired

morphism. In the case $k = 1$ this is just a definition check to see that the definition of K -theory in Definition 2.5 is functorial in \mathcal{C} .

Now consider $k > 1$. In the interest of simplifying the following analysis, we will restrict our attention to the case when $k = 2$; the higher cases follow analogously. The data of a 2-morphism is, for every pair m_1, m_2 , a map of spaces

$$\mu_{m_1, m_2} : K(\mathcal{C}_1)_{m_1} \wedge K(\mathcal{C}_2)_{m_2} \longrightarrow K(\mathcal{C})_{m_1+m_2}.$$

These maps need to be coherent with respect to the spectral structure maps; in particular, we need the following diagram to commute:

$$\begin{array}{ccccc} K(\mathcal{C}_1)_{m_1} \wedge K(\mathcal{C}_2)_{m_2} \wedge S^1 & \xrightarrow{\quad} & K(\mathcal{C}_1)_{m_1} \wedge S^1 \wedge K(\mathcal{C}_2)_{m_2} & & \\ \downarrow \mu_{m_1, m_2} & \searrow & \downarrow & & \downarrow \\ & K(\mathcal{C}_1)_{m_1} \wedge K(\mathcal{C}_2)_{m_2+1} & K(\mathcal{C}_1)_{m_1+1} \wedge K(\mathcal{C}_2)_{m_2} & & \\ & \downarrow \mu_{m_1, m_2+1} & \downarrow \mu_{m_1+1, m_2} & & \\ K(\mathcal{C})_{m_1+m_2} \wedge S^1 & \xrightarrow{\quad} & K(\mathcal{C})_{m_1+m_2+1} & \xleftarrow{\quad} & K(\mathcal{C})_{m_1+1+m_2} \end{array}$$

For a Waldhausen category \mathcal{C} and $0 \leq i \leq n$ we define a functor $\rho_{ni} : \mathcal{C} \rightarrow S_n\mathcal{C}$, which is defined on objects by

$$\rho_{ni}(A)_{jk} = \begin{cases} * & \text{if } j \leq n-i \text{ or } k \geq i, \\ A & \text{otherwise} \end{cases}$$

and extends in the analogous manner to morphisms. Let S^1 be the pointed simplicial set which at level n is equal to the set $\{0, 1, \dots, n\}$; we can also consider S^1 to be a pointed category with only trivial morphisms. Then we have a morphism of simplicial categories $P : \mathcal{C} \times S^1 \rightarrow S\mathcal{C}$ (A, i) $\mapsto \rho_{ni}(A)$.

Lemma 6.3. *P is a well-defined functor of simplicial categories.*

Proof. In order for P to be well-defined we need to show that the image of P is in $S\mathcal{C}$, and that P is compatible with the simplicial maps. The first part of this is true by definition, since ρ_{ni} is constructed to be a valid element of $S_n\mathcal{C}$. For the second part, note that we have

$$\begin{aligned} \partial_j \rho_{ni}(A) &= \begin{cases} \rho_{(n-1)0}(A) & \text{if } j = 0 \text{ and } i = n \text{ or } j = n \text{ and } i = 1, \\ \rho_{(n-1)i}(A) & \text{if } j \leq n-i \text{ and } i \neq n \\ \rho_{(n-1)(i-1)}(A) & \text{if } j > n-i \end{cases} \\ &= \rho_{(n-1)(\partial_j(i))}(A), \end{aligned}$$

where in the right-hand side of the above, $i \in S_n^1$. Analogously,

$$s_j \rho_{ni}(A) = \begin{cases} \rho_{(n+1)i} & \text{if } j \leq n-i \\ \rho_{(n+1)(i+1)} & \text{if } j > n-i \end{cases} = \rho_{(n-1)(s_j(i))}(A),$$

so we are done. \square

We thus have functors

$$P: S^{(m)}\mathcal{C} \times S^1 \longrightarrow S^{(m+1)}\mathcal{C}.$$

By definition, if either $i = 0$ or $A = *$ then $P(A, i) = *$, so P lifts to a map

$$P: NwS^{(m)}\mathcal{C} \wedge S^1 \longrightarrow NwS^{(m+1)}\mathcal{C}.$$

This is the spectral structure map of the K -theory of a symmetric spectrum.

Proof of Proposition 6.2. Consider a biexact functor $F: \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}$. We want to use F to construct morphisms $\mu_{m_1, m_2}: K(\mathcal{C}_1)_{m_1} \wedge K(\mathcal{C}_2)_{m_2} \rightarrow K(\mathcal{C})_{m_1+m_2}$.

The key fact we need about the objects of $S_{\vec{n}_1} \cdots S_{\vec{n}_m} \mathcal{C}$ is that they will be preserved by biexact functors in the following manner. Consider the composition

$$\begin{aligned} S_{\vec{n}_1}^{(m_1)}\mathcal{C}_1 \times S_{\vec{n}_2}^{(m_2)}\mathcal{C}_2 &\longrightarrow [\text{Ar}[\vec{n}_1], \mathcal{C}_1] \times [\text{Ar}[\vec{n}_2], \mathcal{C}_2] \longrightarrow \\ &[\text{Ar}([\vec{n}_1] \times [\vec{n}_2]), \mathcal{C}_2 \times \mathcal{C}_2] \xrightarrow{F \circ} [\text{Ar}([\vec{n}_1] \times [\vec{n}_2]), \mathcal{C}]. \end{aligned}$$

The key extra condition on the objects of $S_{\vec{n}_1 \vec{n}_2}^{(m_1+m_2)}\mathcal{C}$ is that this functor lands in $S_{\vec{n}_1 \vec{n}_2}^{(m_1+m_2)}\mathcal{C}$. By varying the coordinates of \vec{n}_1 and \vec{n}_2 these assemble into exact functors

$$S^{(m_1)}\mathcal{C}_1 \times S^{(m_2)}\mathcal{C}_2 \longrightarrow S^{(m_1+m_2)}\mathcal{C}.$$

Applying $|Nw \cdot|$ to these and noting that any point with the basepoint as one of the coordinates gets mapped to the basepoint, we get maps

$$\mu_{m_1, m_2}: K(\mathcal{C}_1)_{m_1} \wedge K(\mathcal{C}_2)_{m_2} \longrightarrow K(\mathcal{C})_{m_1+m_2}.$$

In order to check these assemble into a map $K(\mathcal{C}_1) \wedge K(\mathcal{C}_2) \rightarrow K(\mathcal{C})$ we that these satisfy the coherence conditions stated earlier. In order to show this, we will show that the following diagram commutes:

$$\begin{array}{ccccc} S^{(m_1)}\mathcal{C}_1 \times S^{(m_2)}\mathcal{C}_2 \times S^1 & \xrightarrow{\quad} & S^{(m_1)}\mathcal{C}_1 \times S^1 \times S^{(m_2)}\mathcal{C}_2 & & \\ \downarrow F & \searrow 1 \times P & & \downarrow P \times 1 & \\ & S^{(m_1)}\mathcal{C}_1 \times S^{(m_2+1)}\mathcal{C}_2 & S^{(m_1+1)}\mathcal{C}_1 \times S^{(m_2)}\mathcal{C}_2 & & \\ & \downarrow F & \downarrow F & & \\ S^{(m_1+m_2)}\mathcal{C} \times S^1 & \xrightarrow{P} & S^{(m_1+m_2+1)}\mathcal{C} & \longleftarrow & S^{(m_1+1+m_2)}\mathcal{C} \end{array}$$

In fact, all of the morphisms except for the two horizontal morphisms are obtained by postcomposing functors $\text{Ar}([\vec{n}_1] \times [\vec{n}_2]) \rightarrow s\mathbf{Cat}$ with P or F . The horizontal morphisms, on the other hand, permute both source and target categories, and then permute the source categories back; everything in between is, once again, postcomposing with P or F . Thus in order for this diagram to commute it suffices to show that the diagram

$$\begin{array}{ccccc}
\mathcal{C}_1 \times \mathcal{C}_2 \times S^1 & \xrightarrow{\quad} & \mathcal{C}_1 \times S^1 \times \mathcal{C}_2 & & \\
\downarrow F \times 1 & \searrow 1 \times P & \downarrow P \times 1 & & \\
& \mathcal{C}_1 \times S.\mathcal{C}_2 & S.\mathcal{C}_1 \times \mathcal{C}_2 & & \\
& \downarrow F & \swarrow F & & \\
\mathcal{C} \times S^1 & \xrightarrow{P} & S.\mathcal{C} & &
\end{array}$$

commutes. Consider a triple $(A_1, A_2, i) \in W_1 \times W_2 \times S_n^1$. To check that the diagram commutes, we need to show that

$$\rho_{ni}(F(A_1, A_2)) = F(A_1, \rho_{ni}(A_2)) = F(\rho_{ni}(A_1), A_2).$$

Looking at each of these at spot jk we have that if $j \leq n - i$ or $k \geq i$, the first is $*$, the second is $F(A_1, *)$ and the third is $F(*, A_2)$, which are all equal because F is biexact. Otherwise, these are all equal to $F(A_1, A_2)$, so are again all equal. So these diagrams commute on objects. Analogously, they commute on all morphisms.

This completes the proof of Proposition 6.2. \square

However, we still have not shown that K is compatible with the closed structure on **WaldCat**. However, as we have just shown that K is in fact a multifunctor, we can consider **WaldCat** to be enriched over **Sp** just by applying K to each of the internal hom-objects to produce a spectrally-enriched category **WaldCat_{Sp}**. We then have the following:

Proposition 6.4. *$K: \mathbf{WaldCat}_{\mathbf{Sp}} \rightarrow \mathbf{Sp}$ is a spectrally-enriched multifunctor.*

Proof. We need to show that K gives a morphism

$$K(\mathbf{WaldCat}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})) \longrightarrow K(\mathcal{D})^{K(\mathcal{C}_1) \wedge \dots \wedge K(\mathcal{C}_k)}.$$

As **Sp** is closed symmetric monoidal, it suffices to show that we get a morphism $K(\mathcal{C}_1) \wedge \dots \wedge K(\mathcal{C}_k) \wedge K(\mathbf{WaldCat}(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})) \rightarrow K(\mathcal{D})$. The evaluation functor defined in Definition 5.3 gives us such a morphism, and it follows directly that this produces the enrichment on K . \square

For more on multifunctors between closed multicategories see [Man12, Section 3].

APPENDIX A. PROOF OF LEMMA 3.4

For ease of reading we restate Lemma 3.4 here.

Lemma 3.4. *Suppose that we are given an indexing category \mathcal{A} along with n subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$. Let $F: \mathcal{A} \rightarrow \mathcal{C}$. Then the southern arrow of the cube I given by*

$$I(\bar{\epsilon}) = \operatorname{colim} F|_{\bigcap_{\epsilon_i=0} \mathcal{A}_i} \quad I(1) = \operatorname{colim} F$$

is an isomorphism.

Proof. We want to show that $\operatorname{colim} I' \cong \operatorname{colim} F$; we will prove this by induction on n .

We begin by proving the inductive step. Suppose that we know that this lemma is true for values lower than n . Let $\mathcal{A}' = \bigcup_{i=1}^{n-1} \mathcal{A}_i$. Let $\mathcal{B} = \mathcal{I}^n \setminus \{(0, 1, \dots, 1), (1, \dots, 1)\}$. By the $n = 2$ case of the lemma applied to I' ,

$$\begin{array}{ccc} \operatorname{colim} I'|_{\mathcal{B} \cap (\mathcal{I}^{n-1} \times \{0\})} & \longrightarrow & \operatorname{colim} I'|_{\mathcal{I}^{n-1} \times \{0\}} \\ \downarrow & & \downarrow \\ \operatorname{colim} I'|_{\mathcal{B}} & \longrightarrow & \operatorname{colim} I' \end{array}$$

is a pushout square. By the $n - 1$ case of the lemma, the upper-left corner is $\operatorname{colim} F|_{\mathcal{A}' \cap \mathcal{A}_n}$, and the lower-left corner is $\operatorname{colim} F|_{\mathcal{A}'}$. The upper-right corner is just $I'(1, \dots, 1, 0) = \operatorname{colim} F|_{\mathcal{A}_n}$. Rewriting this, we see that

$$\begin{array}{ccc} \operatorname{colim} F|_{\mathcal{A}' \cap \mathcal{A}_n} & \longrightarrow & \operatorname{colim} F|_{\mathcal{A}_n} \\ \downarrow & & \downarrow \\ \operatorname{colim} F|_{\mathcal{A}'} & \longrightarrow & \operatorname{colim} I' \end{array}$$

is a pushout square. On the other hand, by the $n = 2$ case of the lemma applied directly to F ,

$$\begin{array}{ccc} \operatorname{colim} F|_{\mathcal{A}' \cap \mathcal{A}_n} & \longrightarrow & \operatorname{colim} F|_{\mathcal{A}_n} \\ \downarrow & & \downarrow \\ \operatorname{colim} F|_{\mathcal{A}'} & \longrightarrow & \operatorname{colim} F \end{array}$$

is also a pushout square. Thus $\operatorname{colim} I' \cong \operatorname{colim} F$, as desired.

The base case is $n = 2$. In particular, we want to show that

$$\begin{array}{ccc} \operatorname{colim} F|_{\mathcal{A}_1 \cap \mathcal{A}_2} & \longrightarrow & \operatorname{colim} F|_{\mathcal{A}_1} \\ \downarrow & & \downarrow \\ \operatorname{colim} F|_{\mathcal{A}_2} & \longrightarrow & \operatorname{colim} F \end{array}$$

is a pushout square. Let X be the pushout of the upper-left part of the square. We have

$$\begin{aligned}
\operatorname{colim} F &\cong \operatorname{coeq} \left(\coprod_{f \in \mathcal{A}} \operatorname{dom} f \rightrightarrows \coprod_{A \in \mathcal{A}} A \right) \\
&\cong \operatorname{colim} \left(\begin{array}{ccc} \coprod_{f \in \mathcal{A}_1 \cap \mathcal{A}_2} \operatorname{dom} f & \xrightleftharpoons[s]{t} & \coprod_{A \in \mathcal{A}_1 \cap \mathcal{A}_2} A \\ L \downarrow & & \downarrow R \\ \coprod_{f \in \mathcal{A}_1 \cup \mathcal{A}_2} \operatorname{dom} f & \xrightleftharpoons[s]{t} & \coprod_{A \in \mathcal{A}_1 \cup \mathcal{A}_2} A \end{array} \right) \\
&\cong \operatorname{coeq} \left(\begin{array}{ccc} \operatorname{coeq} \left(\coprod_{f \in \mathcal{A}_1 \cap \mathcal{A}_2} \operatorname{dom} f \xrightleftharpoons[s]{t} \coprod_{A \in \mathcal{A}_1 \cap \mathcal{A}_2} A \right) & & \\ & L \downarrow \quad \downarrow R & \\ \operatorname{coeq} \left(\coprod_{f \in \mathcal{A}_1 \cup \mathcal{A}_2} \operatorname{dom} f \xrightleftharpoons[s]{t} \coprod_{A \in \mathcal{A}_1 \cup \mathcal{A}_2} A \right) & & \end{array} \right) \\
&\cong \operatorname{coeq} \left(\operatorname{colim} F|_{\mathcal{A}_1 \cap \mathcal{A}_2} \rightrightarrows \operatorname{colim} F|_{\mathcal{A}_1} \amalg \operatorname{colim} F|_{\mathcal{A}_2} \right) \\
&\cong X,
\end{aligned}$$

as claimed. Here, s is the morphism which takes $\operatorname{dom} f$ to itself and t is the morphism given by f on the component indexed by f . L includes $\mathcal{A}_1 \cap \mathcal{A}_2$ into \mathcal{A}_1 , and R includes $\mathcal{A}_1 \cap \mathcal{A}_2$ into \mathcal{A}_2 . □

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